

# CROSSING SPEEDS OF RANDOM WALKS AMONG “SPARSE” OR “SPIKY” BERNOULLI POTENTIALS ON INTEGERS

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**ABSTRACT.** We consider a random walk among i.i.d. obstacles on  $\mathbb{Z}$  under the condition that the walk starts from the origin and reaches a remote location  $y$ . The obstacles are represented by a killing potential, which takes value  $M > 0$  with probability  $p$  and value 0 with probability  $1 - p$ ,  $0 < p < 1$ , independently at each site of  $\mathbb{Z}$ . We consider the walk under both quenched and annealed measures. It is known that under either measure the crossing time from 0 to  $y$  of such walk,  $\tau_y$ , grows linearly in  $y$ . More precisely, the expectation of  $\tau_y/y$  converges to a limit as  $y \rightarrow \infty$ . The reciprocal of this limit is called the asymptotic speed of the conditioned walk. We study the behavior of the asymptotic speed in two regimes: (1) as  $p \rightarrow 0$  for  $M$  fixed (“sparse”), and (2) as  $M \rightarrow \infty$  for  $p$  fixed (“spiky”). We observe and quantify a dramatic difference between the quenched and annealed settings.

## 1. INTRODUCTION

We shall start with a model description and the necessary notation. After stating our results we offer an informal discussion, references, and describe a conjecture which makes out results a part of a more general picture.

**Model description and main results.** Let  $V(x, \omega)$ ,  $x \in \mathbb{Z}$ , be i.i.d. random variables on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  such that

$$(1.1) \quad \mathbb{P}(V(0, \cdot) \geq 0) = 1, \quad \mathbb{P}(V(0, \cdot) > 0) > 0, \quad \text{and} \quad \mathbb{E}[V(0, \cdot)] < \infty.$$

These random variables represent a random potential on  $\mathbb{Z}$ . Given a realization of the potential, i.e. for each fixed  $\omega \in \Omega$ , we consider a Markov chain on  $\mathbb{Z} \cup \{\dagger\}$  with transition probabilities:

$$(1.2) \quad p(x, y, \omega) = \begin{cases} 1 - e^{-V(x, \omega)}, & \text{if } y = \dagger, x \neq \dagger; \\ \frac{1}{2} e^{-V(x, \omega)}, & \text{if } y = x \pm 1, x \neq \dagger; \\ 1, & \text{if } y = \dagger, x = \dagger; \\ 0, & \text{otherwise.} \end{cases}$$

Informally, this is a so called “killed random walk”: at each site  $x$  the walk either gets killed with probability  $1 - e^{-V(x, \omega)}$  (and moves to the absorbing state  $\dagger$ ) or survives and moves to one of the two neighboring sites with equal probabilities  $e^{-V(x, \omega)}/2$ . The corresponding path measure and the expectation with respect to it will be denoted by  $P^\omega$  and  $E^\omega$  respectively. Unless stated otherwise, the killed random walk starts from 0.

Denote by  $(S_n)_{n \geq 0}$  a path on  $\mathbb{Z} \cup \{\dagger\}$  and set

$$(1.3) \quad \tau_x := \inf\{n > 0 : S_n = x\} \in \mathbb{N} \cup \{\infty\}.$$

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We shall put different measures on the set of nearest neighbor paths. Measure  $P$ , and the corresponding expectation  $E$ , will always refer to the simple symmetric random walk on  $\mathbb{Z}$  starting from 0.

Next, fix  $y > 0$  and consider the conditional measure on nearest neighbor paths starting from 0, which is defined by

$$(1.4) \quad Q_y^\omega(\cdot) := P^\omega(\cdot | \tau_y < \infty).$$

Measure  $Q_y^\omega$  is called the quenched path measure. Notice that it can be equivalently defined as follows:

$$(1.5) \quad Q_y^\omega(A) := (Z_y^\omega)^{-1} E \left( \mathbb{1}_{\{\tau_y < \infty\}} \mathbb{1}_A e^{-\sum_{n=0}^{\tau_y-1} V(S_n, \omega)} \right), \text{ where}$$

$$(1.6) \quad Z_y^\omega := P^\omega(\tau_y < \infty) = E \left( \mathbb{1}_{\{\tau_y < \infty\}} e^{-\sum_{n=0}^{\tau_y-1} V(S_n, \omega)} \right).$$

Finally, we define the annealed measure  $Q_y$ , which is a measure on the product space of  $\Omega$  and the nearest-neighbor paths starting from 0. We let

$$(1.7) \quad Q_y(B) := \frac{\mathbb{E}(P^\omega(B \cap \{\tau_y < \infty\}))}{\mathbb{E}(P^\omega(\tau_y < \infty))}.$$

Whenever the starting point of a process is different from 0, we shall indicate it with a superscript, for example,  $Q_y^{\omega, x}$  will denote a quenched path measure of a killed random walk, which starts at  $x$  and is conditioned to hit  $y$ ,  $y > x$ .

The quenched and annealed asymptotic speeds,  $v_V^{\text{que}}$  and  $v_V^{\text{ann}}$ , are the deterministic quantities, defined by

$$(1.8) \quad \frac{1}{v_V^{\text{que}}} := \lim_{y \rightarrow \infty} \frac{E_{Q_y^\omega}(\tau_y)}{y} \quad (\mathbb{P}\text{-a.s.}); \quad \frac{1}{v_V^{\text{ann}}} := \lim_{y \rightarrow \infty} \frac{E_{Q_y}(\tau_y)}{y}.$$

By Proposition 2.1 below and [KM12, Theorem 1.2] respectively these limits exist and are finite.

In this paper we consider i.i.d. Bernoulli potentials  $V_{p,M}(x, \cdot)$ ,  $x \in \mathbb{Z}$ ,  $p \in (0, 1)$ ,  $M > 0$ ,

$$(1.9) \quad \mathbb{P}(V_{p,M}(x, \cdot) = M) = 1 - \mathbb{P}(V_{p,M}(x, \cdot) = 0) = p,$$

and study the behavior of the corresponding quenched and annealed asymptotic speeds  $v_{p,M}^{\text{que}}$  and  $v_{p,M}^{\text{ann}}$  in two regimes: “sparse” ( $p \rightarrow 0$ ,  $M$  is fixed) and “spiky” ( $M \rightarrow \infty$ ,  $p$  is fixed). Our main results are contained in the following two theorems. By  $f(x) \sim g(x)$  as  $x \rightarrow \alpha$  we mean that  $\lim_{x \rightarrow \alpha} f(x)/g(x) = 1$ .

**Theorem 1.1.** *With  $V_{p,M}$  as above, the quenched speed,  $v_{p,M}^{\text{que}}$ , satisfies*

$$(1.10) \quad v_{p,M}^{\text{que}} \sim \frac{3p}{2} \text{ as } p \rightarrow 0 \text{ uniformly in } M \in [M_0, \infty), \forall M_0 > 0;$$

$$(1.11) \quad v_{p,M}^{\text{que}} \sim \frac{3p}{2-p+2p^2} \text{ as } M \rightarrow \infty \text{ locally uniformly in } p \in (0, 1].$$

**Theorem 1.2.** *With  $V_{p,M}$  as above, the annealed speed,  $v_{p,M}^{\text{ann}}$ , satisfies*

$$(1.12) \quad -\log v_{p,M}^{\text{ann}} \sim \frac{2(e^M - 1)}{p} \text{ as } p \rightarrow 0 \text{ uniformly in } M \in [M_0, \infty), \forall M_0 > 0;$$

$$(1.13) \quad -\log v_{p,M}^{\text{ann}} \sim \frac{2(1-p)e^M}{p} \text{ as } M \rightarrow \infty \text{ locally uniformly in } p \in (0, 1).$$

**Remark 1.3.** Observe that in the “sparse” regime both the quenched and annealed speeds vanish but the latter does so at a dramatically higher rate. In the “spiky” regime the quenched speed converges to a positive constant while the annealed one vanishes extremely fast. This striking difference is a purely one-dimensional phenomenon. In dimensions two and higher the existence of the annealed asymptotic velocity is known ([IV12b, Theorem C]) but the existence of the quenched asymptotic velocity is still a largely open problem (see [IV12a, Section 1.2] and references therein). Thus the comparison question might seem a bit premature. Nevertheless, even when both speeds are well-defined, we do not expect to see anything like this in higher dimensions.

**Discussion and an open problem.** There are many papers concerning the relationship of the quenched and annealed Lyapunov exponents of a random walk in a random potential on  $\mathbb{Z}^d$ ,  $d \geq 1$  (see [Fl08], [Zy09], [KMZ11], [IV12c], [Zy12], and references therein). Lyapunov exponents represent the exponential decay rates of the quenched and annealed survival probabilities,  $P^\omega(\tau_y < \infty)$  and  $\mathbb{E}(P^\omega(\tau_y < \infty))$  respectively. The equality or non-equality of quenched and annealed Lyapunov exponents determines whether the disorder introduced by the random environment is “weak” or “strong”. According to this classification any non-trivial disorder in the one-dimensional case is strong (as well as in dimensions 2 and 3 under mild additional conditions on  $V$ , see Theorem 1 and a paragraph after it in [Zy12]). Our results provide more refined information about differences between quenched and annealed behavior.

In dimensions 4 and higher one expects a transition from weak to strong disorder for i.i.d. potentials of the form  $\gamma V$ ,  $V \geq 0$ , for some  $\gamma^* \in (0, \infty)$ . Such result is known to hold when  $V$  is bounded away from 0 ([Fl08], [Zy09], [IV12c]).

Let us discuss the case of a “small” potential, i.e. the potential of the form  $\gamma V$  where  $\gamma \ll 1$ , in more detail. We shall restrict ourselves to dimension 1 but we believe that a similar result holds in all dimensions (when the quenched speed is well defined, see Remark 1.3 above).

Let  $(V(x, \cdot), x \in \mathbb{Z})$  be i.i.d. random variables satisfying (1.1) and  $v_V^{\text{que}}$  and  $v_V^{\text{ann}}$  be the quenched and annealed speeds as defined in (1.8). Using our methods it should be not difficult to show that as  $\gamma \downarrow 0$

$$(1.14) \quad v_{\gamma V}^{\text{que}}, v_{\gamma V}^{\text{ann}} \sim \sqrt{2\gamma \mathbb{E}(V)}.$$

This result would complement the results of this paper. The relation (1.14) is suggested by the following two facts.

(i) The speeds can be equivalently defined as follows (Proposition 2.1 below and [KM12, Theorem 1.2]):

$$(1.15) \quad \frac{1}{v_V^{\text{que}}} = \frac{d}{d\lambda} \alpha_{\lambda+V}(1) \Big|_{\lambda=0+}, \quad \frac{1}{v_V^{\text{ann}}} = \frac{d}{d\lambda} \beta_{\lambda+V}(1) \Big|_{\lambda=0+},$$

where for each  $\lambda \geq 0$  the non-random quantities

$$(1.16) \quad \alpha_{\lambda+V}(1) := - \lim_{y \rightarrow \infty} \frac{1}{y} \log E \left( \mathbb{1}_{\{\tau_y < \infty\}} e^{-\sum_{n=0}^{\tau_y-1} (\lambda + V(S_n, \omega))} \right) \quad (\mathbb{P}\text{-a.s.});$$

$$(1.17) \quad \beta_{\lambda+V}(1) := - \lim_{y \rightarrow \infty} \frac{1}{y} \log \mathbb{E} \left[ E \left( \mathbb{1}_{\{\tau_y < \infty\}} e^{-\sum_{n=0}^{\tau_y-1} (\lambda + V(S_n, \omega))} \right) \right]$$

are the quenched and annealed (respectively) Lyapunov exponents of a random walk in the potential  $\lambda + V$ . Under our assumptions, both limits are positive and finite. For more details about the existence and properties of Lyapunov exponents see [Ze98], [Fl07], and [Mou12].

(ii) It was shown in [KMZ11] (see also [Wa01], [Wa02]) that  $\alpha_{\gamma V}(1)$ ,  $\beta_{\gamma V}(1) \sim \sqrt{2\gamma\mathbb{E}(V)}$  as  $\gamma \downarrow 0$ . Since

$$\alpha_{\gamma(\lambda+V)}(1) \sim \sqrt{2\gamma(\lambda + \mathbb{E}(V))} \quad \text{as } \gamma \downarrow 0 \quad \text{and} \quad \frac{d}{d(\gamma\lambda)} = \frac{1}{\gamma} \frac{d}{d\lambda},$$

a formal differentiation of  $\sqrt{2\gamma(\lambda + \mathbb{E}(V))}$  with respect to  $\lambda$  and division by  $\gamma$  lead to the conjecture (1.14). We remark that in the case of a constant potential (without loss of generality we set  $V \equiv 1$ ) it is easy to compute the Lyapunov exponent explicitly:  $\alpha_{\lambda+\gamma}(1) = \beta_{\lambda+\gamma}(1) = \log(e^{\lambda+\gamma} + \sqrt{e^{2(\lambda+\gamma)} - 1})$ ,  $\gamma, \lambda \geq 0$ , and obtain from (1.15)

$$v_\gamma := v_\gamma^{\text{que}} = v_\gamma^{\text{ann}} = 1 - \frac{1}{e^{2\gamma}(\sqrt{1 - e^{-2\gamma}} + 1)}, \quad \gamma \geq 0.$$

Thus,  $\alpha_\gamma(1) = \beta_\gamma(1) \sim \sqrt{2\gamma}$  and  $v_\gamma \sim \sqrt{2\gamma}$  as  $\gamma \downarrow 0$ . The latter is (1.14) in this special case.

The relation (1.14) further supports an informal statement that when the potential  $V$  is “small” (but not “sparse”!) both the quenched and annealed behavior in such a potential are well approximated by the behavior of the walk in a constant potential  $\mathbb{E}(V)$ .

*An open problem.* Theorems 1.1, 1.2, and asymptotics (1.14) provide information about  $v_{p,M}^{\text{que}}$  and  $v_{p,M}^{\text{ann}}$  when  $M \rightarrow \infty$  or  $p \rightarrow 0$  or  $M \rightarrow 0$ . A more interesting and challenging question is to study surfaces formed by the speeds when  $(p, M) \in [0, 1] \times [0, \infty)$ .

As the first step, fix  $p \in (0, 1)$  and consider  $v_{p,M}^{\text{que}}$  and  $v_{p,M}^{\text{ann}}$  as functions of  $M$ . It does not seem surprising that the annealed environment will typically be more sparse than the quenched one and thus the random walker will feel less need to quickly navigate towards the goal  $y$ . Heuristically, one might expect that  $v_{p,M}^{\text{que}}$  is strictly increasing in  $M$  for a fixed  $p$  and that  $v_{p,M}^{\text{ann}}$  has a single maximum as  $M$  changes from 0 to  $\infty$  (see Figure 1). Is this really the

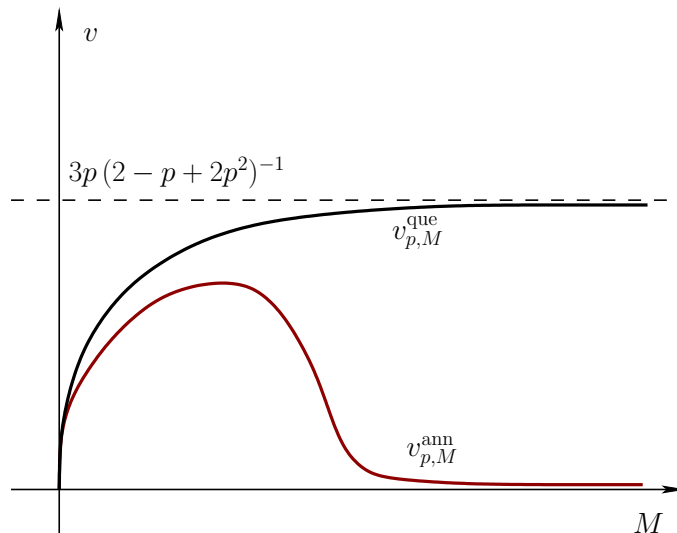


FIGURE 1. Conjectured shape of  $v_{p,M}^{\text{que}}$  and  $v_{p,M}^{\text{ann}}$  as functions of  $M$  for a fixed  $p \in (0, 1)$ . By (1.14),  $v_{p,M}^{\text{que}}, v_{p,M}^{\text{ann}} \sim \sqrt{2pM}$  as  $M \rightarrow 0$ .

case? These questions are presumably hard to resolve analytically. We are not aware even of any simulations done on this problem (see though [MG02] as well as [GP82]).

**Organization of the paper.** In Section 2 we prove Theorem 1.1. The proof of Theorem 1.2 is given in Section 3, which is subdivided into 3 subsections. Subsection 3.1 gives heuristics and a short outline of the proof. Subsection 3.2 is the technical core of the proof. There we study the environment under the annealed measure. Subsection 3.3 uses the estimates obtained in the previous subsection and completes the proof of Theorem 1.2. The Appendix contains several auxiliary results and proofs of several lemmas used in the main part of the paper.

**Terminology.** Sites at which the potential is equal to  $M$  will be called *occupied* sites. *Vacant* sites are unoccupied sites. An interval is *empty* if all its sites are unoccupied. We reserve the term vacant interval for maximal empty intervals.

## 2. QUENCHED SPEED

The results of Theorem 1.1 are rather straightforward. The fact that the speed is at most of order  $p$  comes immediately from the fact that no matter what the value of  $M$ , the time for a conditioned random walk to traverse an empty interval of size  $R$  will be of order  $R^2$ , corresponding to speed of order  $1/R$ . The upper bound on speed follows since most sites lie in vacant intervals of size of order  $1/p$ .

We shall need two facts. The first is a very basic fact about the standard random walk but we do not have a reference at hand and, thus, give a proof in the Appendix.

**Proposition 2.1.** *Let  $(S_j)_{j \geq 0}$  be the simple symmetric random walk,  $n \in \mathbb{N}$ , and  $S_0 = k \in \{1, 2, \dots, n-1\}$ . Then*

$$E^k(\tau_n; \tau_n < \tau_0) = \frac{k(n-k)(n+k)}{3n}.$$

The second is mostly a consequence of the ergodic theorem (see [Sz94, (1.30) and Theorem 2.6] for a treatment of Brownian motion among Poissonian obstacles). The proof is given in the Appendix.

**Proposition 2.2.** *Let  $V(x, \cdot)$ ,  $x \in \mathbb{Z}$  be i.i.d. random variables, which satisfy (1.1). Then there exists limit*

$$\begin{aligned} \frac{1}{v^{\text{que}}} &:= \lim_{y \rightarrow \infty} \frac{E_{Q_y^\omega}(\tau_y)}{y} = \mathbb{E}(E^\omega(\tau_1 | \tau_1 < \infty)) < \infty \quad (\mathbb{P}\text{-a.s.}), \quad \text{and, moreover,} \\ \frac{1}{v^{\text{que}}} &= \frac{d}{d\lambda} \alpha_{\lambda+V}(1) \Big|_{\lambda=0+}. \end{aligned}$$

The key idea for calculation of the quenched speed is an observation that the main contribution to  $\mathbb{E}[E^\omega(\tau_1 | \tau_1 < \infty)]$  comes from paths which hit  $\tau_1$  before entering  $(-\infty, -a_1]$  where  $-a_1 = \max\{x \leq 0 : V(x, \cdot) = M\}$  at some positive time. Our first step is to compute the main term.

**Lemma 2.3.** *For every  $a \in \mathbb{N} \cup \{0\}$*

$$\mathbb{E}[E^\omega(\tau_1 | \tau_1 < \tau_{-a_1})] = \frac{2-p+2p^2}{3p}.$$

*Proof.* Notice that our definition of  $\tau_0$  (see (1.3)) implies  $E^\omega(\tau_1 | \tau_1 < \tau_0) = 1$ . By Proposition 2.2 we have

$$\mathbb{E}[E^\omega(\tau_1 | \tau_1 < \tau_{-a_1})] = p + p \sum_{a=1}^{\infty} \frac{2a+1}{3} (1-p)^a = p + \frac{1-p}{3} \left( \frac{2}{p} + 1 \right) = \frac{2-p+2p^2}{3p}.$$

□

Given  $\omega$ , let  $-a_1 > -a_2 > \dots$  be the occupied sites in  $(-\infty, 0]$ ,  $I_j = [-a_{j+1}, -a_j]$ ,  $j \in \mathbb{N}$ , and  $I = [-a_1, 1)$ . Then

$$(2.1) \quad \mathbb{E}[E^\omega(\tau_1 | \tau_1 < \infty)] = \mathbb{E}[E^\omega(\tau_1 \mathbb{1}_{\{\tau_1 < \tau_{-a_1}\}} | \tau_1 < \infty)] \\ + \mathbb{E}[E^\omega(\mathbb{1}_{\{\tau_1 > \tau_{-a_1}\}} \sum_{n=0}^{\tau_1-1} \mathbb{1}_{\{S_n \in I\}} | \tau_1 < \infty)] + \sum_{j=1}^{\infty} \mathbb{E}[E^\omega(\mathbb{1}_{\{\tau_1 > \tau_{-a_1}\}} \sum_{n=0}^{\tau_1-1} \mathbb{1}_{\{S_n \in I_j\}} | \tau_1 < \infty)].$$

Observe also that the first term in the right hand side of (2.1) equals

$$(2.2) \quad \mathbb{E}[E^\omega(\tau_1 | \tau_1 < \tau_{-a_1}) P^{\omega,0}(\tau_1 < \tau_{-a_1} | \tau_1 < \infty)] = \\ \mathbb{E}[E^\omega(\tau_1 | \tau_1 < \tau_{-a_1})] - \mathbb{E}[E^\omega(\tau_1 | \tau_1 < \tau_{-a_1}) P^{\omega,0}(\tau_1 > \tau_{-a_1} | \tau_1 < \infty)].$$

We shall need the following three elementary lemmas.

**Lemma 2.4.** *For  $a_1 \in \mathbb{N} \cup \{0\}$*

$$P^{\omega,0}(\tau_1 > \tau_{-a_1} | \tau_1 < \infty) \leq e^{-M} \wedge ((2a_1(e^M - 1) + 1)(1 + a_1))^{-1}.$$

**Lemma 2.5.** *There is a constant  $C_1$  such that*

$$E^\omega(\mathbb{1}_{\{\tau_1 > \tau_{-a_1}\}} \sum_{n=0}^{\tau_1-1} \mathbb{1}_{\{S_n \in I\}} | \tau_1 < \infty) \leq C_1(e^M - 1)^{-1}.$$

**Lemma 2.6.** *There is a constant  $C_2$  such that for every  $j \in \mathbb{N}$*

$$E^\omega(\mathbb{1}_{\{\tau_1 > \tau_{-a_1}\}} \sum_{n=0}^{\tau_1-1} \mathbb{1}_{\{S_n \in I_j\}} | \tau_1 < \infty) \leq \frac{C_2 e^{-Mj}}{(1 - e^{-M})^2} \frac{|I_j|}{|I|^2},$$

where  $|A|$  is the Lebesgue measure of the set  $A$ .

Let us assume these facts (see Appendix for proofs) and derive Theorem 1.1.

*Proof of Theorem 1.1.* The right hand side of the inequality in Lemma 2.4 does not exceed  $e^{-M}/(1 + a_1)$ . Therefore,

$$\mathbb{E}[E^\omega(\tau_1 | \tau_1 < \tau_{-a_1}) P^{\omega,0}(\tau_1 > \tau_{-a_1} | \tau_1 < \infty)] \leq \\ e^{-M} + p e^{-M} \sum_{a_1=1}^{\infty} (1-p)^{a_1} \frac{2a_1+1}{3(a_1+1)} < e^{-M} + \frac{2p e^{-M}}{3} \sum_{a_1=1}^{\infty} (1-p)^{a_1} \leq \frac{5}{3} e^{-M}.$$

This immediately gives

$$\lim_{p \rightarrow 0} p \mathbb{E}[E^\omega(\tau_1 | \tau_1 < \tau_{-a_1}) P^{\omega,0}(\tau_1 > \tau_{-a_1} | \tau_1 < \infty)] = 0; \\ \lim_{M \rightarrow \infty} \mathbb{E}[E^\omega(\tau_1 | \tau_1 < \tau_{-a_1}) P^{\omega,0}(\tau_1 > \tau_{-a_1} | \tau_1 < \infty)] = 0.$$

Lemma 2.5 takes care of the second term in the right hand side of (2.1). By Lemma 2.6 and the independence of the values of the potential at distinct sites, the last term in (2.1) is

bounded by (we define the function  $(1-p)^{-1} \ln(1/p)$  to be 1 at  $p = 1$  by continuity)

$$\begin{aligned} \frac{C_2}{(1-e^{-M})^2} \sum_{j=1}^{\infty} e^{-Mj} \mathbb{E}[(a_{j+1} - a_j)] \mathbb{E}[(a_1 + 1)^{-1}] \\ = \frac{C_2}{(1-e^{-M})^2} \sum_{j=1}^{\infty} e^{-Mj} \frac{\ln(1/p)}{1-p} = \frac{C_2 e^{-M} \ln(1/p)}{(1-p)(1-e^{-M})^3}. \end{aligned}$$

This expression clearly vanishes as  $M \rightarrow \infty$  locally uniformly in  $p \in (0, 1]$ . After multiplication by  $p$  it converges to 0 as  $p \rightarrow 0$  uniformly on every interval  $[M_0, \infty)$ . The only term left in the right hand side of (2.1) and (2.2) is the main term,  $\mathbb{E}[E^\omega(\tau_1 | \tau_1 < \tau_{-a_1})]$ , which has the claimed asymptotics by Lemma 2.3.  $\square$

### 3. ANNEALED SPEED

We start by introducing additional notation. Let

$$Q_{0,y}(\cdot) := Z_{0,y}^{-1} \mathbb{E}[P^\omega(\cdot; \tau_y < \tau_0, \tau_y < \infty)], \quad Z_{0,y} := \mathbb{E}[P^\omega(\tau_y < \tau_0, \tau_y < \infty)].$$

The corresponding quenched path measure  $Q_{0,y}^\omega$  is given by

$$Q_{0,y}^\omega(\cdot) = (Z_{0,y}^\omega)^{-1} P^\omega(\cdot; \tau_y < \tau_0, \tau_y < \infty), \quad Z_{0,y}^\omega := P^\omega(\tau_y < \tau_0, \tau_y < \infty).$$

**3.1. Heuristics and goals.** Our first observation is that we can replace the measure  $Q_y$  with  $Q_{0,y}$  (see Proposition 3.1, (3.4), and the proof of (1.7) in [KM12]). The key ingredient of the proof of Theorem 1.2 is the study of environments under  $Q_{0,y}$ . We show that  $Q_{0,y}$  (for large  $y$ ) considered as measure on environments is “comparable” to the product of Bernoulli measures with  $p$  replaced by  $p_{\text{ann}}$ , where for fixed  $M$  and small  $p$  or fixed  $p$  and large  $M$

$$(3.1) \quad -\log p_{\text{ann}} \sim K(p, M) := 2p^{-1}(1-p)(e^M - 1).$$

For i.i.d. Bernoulli potentials we already have the result that the quenched speed is proportional to the probability of having a non-zero potential at a given site, which is now  $p_{\text{ann}}$ . This observation together with (3.1) leads to the limits (1.12) and (1.13).

We shall give a detailed proof of (1.12). The proof of (1.13) is very similar but easier and is omitted but we shall write all steps in such a way that they can be easily adapted to the case when  $M \rightarrow \infty$  and  $p$  is fixed.

Our goal will be to construct subsets of environments that are essential and on which the walk has the claimed speed behavior. More precisely, to obtain a lower bound on  $-\log v_{p,M}^{\text{ann}}$  we shall restrict  $Q_{0,y}$  to environments  $\Omega_y^1 = \Omega_y^1(\varepsilon, p, M)$  with the following properties: for every  $\varepsilon > 0$  there is  $p_0 = p_0(\varepsilon)$  such that for each  $p < p_0$  there is  $y_0 = y_0(p)$  such that for all  $y > y_0$

(L1)  $Q_{0,y}(\Omega_y^1) \geq 1/2$  and

(L2) for every  $\omega \in \Omega_y^1$

$$E_{Q_{0,y}^\omega}(\tau_y) \geq C_1 y e^{(1-\varepsilon)K(p,M)},$$

where  $C_1$  does not depend on  $y, \omega, p$ . Then

$$\frac{E_{Q_{0,y}}(\tau_y)}{y} \geq (y Z_{0,y})^{-1} \mathbb{E}(E_{Q_{0,y}^\omega}(\tau_y) Z_{0,y}^\omega; \Omega_y^1) \geq C_1 e^{(1-\varepsilon)K(p,M)} Q_y(\Omega_y^1) \geq \frac{C_1}{2} e^{(1-\varepsilon)K(p,M)},$$

and

$$(3.2) \quad -\liminf_{p \rightarrow 0} p \log v_{p,M}^{ann} \geq (1 - \varepsilon) \lim_{p \rightarrow 0} pK(p, M) = 2(1 - \varepsilon)(e^M - 1).$$

For the upper bound we shall consider environments  $\Omega_y^2 = \Omega_y^2(\varepsilon, p, M)$  for which the following hold: for every  $\varepsilon > 0$  there is  $p_0 = p_0(\varepsilon)$  such that for each  $p < p_0$

$$(U1) \quad \lim_{y \rightarrow \infty} y Q_{0,y}(\Omega \setminus \Omega_y^2) = 0 \text{ and}$$

$$(U2) \quad \text{there is } y_0 = y_0(p) \text{ such that for all } y > y_0, \omega \in \Omega_y^2$$

$$E_{Q_{0,y}^\omega}(\tau_y) \leq C_2 y e^{(1+\varepsilon)K(p,M)},$$

where  $C_2$  does not depend on  $y, \omega, p$ . Then

$$\begin{aligned} \frac{E_{Q_{0,y}}(\tau_y)}{y} &\leq (y Z_{0,y})^{-1} \left( \mathbb{E}(E_{Q_{0,y}^\omega}(\tau_y) Z_{0,y}^\omega; \Omega_y^2) + \mathbb{E}(E_{Q_{0,y}^\omega}(\tau_y) Z_{0,y}^\omega; \Omega \setminus \Omega_y^2) \right) \\ &\leq C_2 e^{(1+\varepsilon)K(p,M)} + (y Z_{0,y})^{-1} \mathbb{E}(E_{Q_{0,y}^\omega}(\tau_y) Z_{0,y}^\omega; \Omega \setminus \Omega_y^2). \end{aligned}$$

By Lemma A.1 (see Appendix),  $E_{Q_{0,y}^\omega}(\tau_y) \leq 3y^2$ . Combining this with (U1) we get

$$(3.3) \quad -\limsup_{p \rightarrow 0} p \log v_{p,M}^{ann} \leq (1 + \varepsilon) \lim_{p \rightarrow 0} pK(p, M) = 2(1 + \varepsilon)(e^M - 1).$$

Since  $\varepsilon$  is arbitrary, relations (3.2) and (3.3) imply (1.12). Our task will be to construct  $\Omega_y^i$ ,  $i = 1, 2$ , with the desired properties. This will be done in Subsection 3.3 after we obtain information about a typical environment under the annealed measure  $Q_{0,y}$ . The latter is the content of the next subsection.

### 3.2. Environment under the annealed measure.

**Lemma 3.1.** *Let  $x_0 := 0 < x_1 < \dots < x_n$ ,  $r_i = x_i - x_{i-1}$ ,  $i = 1, 2, \dots, n$ , and consider an environment such that  $\{x_1, x_2, \dots, x_n\}$ ,  $n \in \mathbb{N}$ , is the set of all occupied sites in  $(0, x_n]$ . Denote by  $u_n$  the probability that a random walk starting at  $x_{n-1}$  reaches  $x_n$  before hitting 0, i.e.  $u_n = P^{x_{n-1}, \omega}(\tau_{x_n} < \tau_0)$ ,  $n \in \mathbb{N}$ . Then*

$$(3.4) \quad u_1 = \frac{e^{-V(0, \omega)}}{2r_1} = e^{M-V(0, \omega)} F_M(0, r_1, 0); \quad u_n = F_M(r_{n-1}, r_n, u_{n-1}), \quad n > 1,$$

where  $F : (\mathbb{N} \cup \{0\}) \times \mathbb{N} \times [0, 1] \rightarrow [0, 1]$ ,

$$(3.5) \quad F_M(\ell, r, u) = \begin{cases} \frac{e^{-M}}{2r} \left( 1 - e^{-M} \left( 1 - \frac{1}{2r} - \frac{1-u}{2\ell} \right) \right)^{-1}, & \text{if } \ell \neq 0; \\ \frac{e^{-M}}{2r}, & \text{if } \ell = 0. \end{cases}$$

The proof is given in the Appendix.

**Corollary 3.2.** *As  $\ell, r \rightarrow \infty$*

$$(3.6) \quad F_M(\ell, r, u) \sim \frac{e^{-M}}{2r(1 - e^{-M})}, \quad \text{uniformly in } u \in [0, 1].$$



From now on we shall identify every environment  $\omega$  on  $(0, y)$  with the vector  $\bar{R}_{N+1} = (R_1, R_2, \dots, R_{N+1}) \in \cup_{i=1}^{\infty} \mathbb{N}^i$  of successive distances between  $N = N(y, \omega)$  occupied sites in  $(0, y)$ , where  $R_1$  is the distance from the first positive occupied site in  $(0, y)$  to the origin and  $R_{N+1}$  is the distance from the last occupied site in  $(0, y)$  to  $y$ . If the interval  $(0, y)$  is empty then we set  $N = 0$  and  $R_1 = y$ .

**Corollary 3.3.** *For any  $n \in \{0, 1, 2, \dots, y-1\}$  and  $(r_1, r_2, \dots, r_{n+1}) \in \mathbb{N}^{n+1}$  with  $\sum_{i=1}^{n+1} r_i = y$ ,  $r_0 = 0$ ,  $u_0 = 0$ ,*

$$(3.7) \quad \begin{aligned} Q_{0,y}(N = n, \bar{R}_{n+1} = (r_1, r_2, \dots, r_{n+1})) \\ = Z_{0,y}^{-1} \left( \frac{e^M(1-p)}{p} + 1 \right) \times \prod_{i=1}^{n+1} (p(1-p)^{r_i-1} F_M(r_{i-1}, r_i, u_{i-1})) \\ = Z_{0,y}^{-1} (1-p)^y \left( \frac{e^M}{\varrho} + 1 \right) \times \prod_{i=1}^{n+1} (\varrho F_M(r_{i-1}, r_i, u_{i-1})) \end{aligned}$$

where  $\varrho := p/(1-p)$ .

All information about the dependence of (3.7) on  $(r_1, r_2, \dots, r_{n+1})$  is contained in the last product. To study its behavior, we consider the probability  $U_n$  that the killed random walk reaches the  $n$ -th occupied site prior to the first return to 0 if  $(R_i)_{i \in \mathbb{N}}$  are i.i.d. positive integer-valued random variables with probability mass function  $q(1-q)^r$ ,  $r \in \mathbb{N}$ , for some  $q \in (0, 1)$ . We shall assume that 0 is occupied. Then (setting  $r_0 = 0$ ,  $u_0 = 0$ )

$$U_n = \sum_{r_1, r_2, \dots, r_n \in \mathbb{N}} \prod_{i=1}^n G_q(r_i) F_M(r_{i-1}, r_i, u_{i-1}).$$

**Lemma 3.4.** *For every  $q \in (0, 1)$  there exists  $\mu(q)$ ,*

$$(3.8) \quad \mu : (0, 1) \mapsto \left( \frac{q \log(1/q)}{2e^M(1-q)}, \frac{q \log(1/q)}{2(e^M - 1)(1-q)} \right), \quad \mu \in C(0, 1),$$

such that  $\mu^n(1 - e^{-M}) \leq U_n \leq \mu^n$  for all  $n \in \mathbb{N}$ .

*Proof.* It is obvious that  $U_n \geq U_m U_{n-m}$  for  $1 \leq m \leq n$ . This implies that the sequence  $\log U_n$ ,  $n \in \mathbb{N}$ , is superadditive, and, thus,

$$(3.9) \quad \lim_{n \rightarrow \infty} \frac{\log U_n}{n} = \sup_n \frac{\log U_n}{n} =: \log \mu.$$

Therefore,  $U_n \leq \mu^n$  for all  $n \in \mathbb{N}$ .

For the lower bound, consider a killed random walk, which starts from the origin in an environment, such that all sites to the left from 0 are empty. Let  $\widetilde{U}_n$  be the probability that this walk reaches the  $n$ -th occupied site in  $(0, \infty)$ . Conditioning on the number of returns to the origin before reaching the  $n$ -th occupied site, we obtain

$$(3.10) \quad U_n \leq \widetilde{U}_n \leq \sum_{k=0}^{\infty} U_n e^{-Mk} = \frac{U_n}{1 - e^{-M}}.$$

Notice that the sequence  $\log \widetilde{U}_n$ ,  $n \in \mathbb{N}$ , is subadditive. From this, (3.10), and (3.9) it follows that

$$\log \mu = \lim_{n \rightarrow \infty} \frac{\log \widetilde{U}_n}{n} = \inf_n \frac{\log \widetilde{U}_n}{n}.$$

We conclude that  $\widetilde{U}_n \geq \mu^n$ . By (3.10),  $U_n \geq \mu^n(1 - e^{-M})$  for all  $n \in \mathbb{N}$ .

Properties of  $\mu$  follow from the inequality  $U_n^{1/n} \leq \mu \leq U_n^{1/n}(1 - e^{-M})^{-1/n}$ ,  $n \in \mathbb{N}$ . Taking  $n = 1$  we find

$$U_1 = \sum_{r=1}^{\infty} q(1-q)^{r-1}/(2e^M r) = \frac{q \log(1/q)}{2e^M(1-q)}$$

and obtain the desired bounds on  $\mu$ . Continuity of  $\mu$  follows from continuity of  $U_n^{1/n}$  as a function of  $q$  for each  $n$  and the fact that  $(1 - e^{-M})^{-1/n} \rightarrow 1$  as  $n \rightarrow \infty$ .  $\square$

**Corollary 3.5.** *For each  $\varrho \in (0, \infty)$  there is  $q = q(\varrho, M) \in (0, 1)$  such that*

$$(3.11) \quad 2(1 - e^{-M}) \leq e^{-M} \varrho \log(1/q) \leq 2,$$

and for all  $n \in \mathbb{N}$

$$(3.12) \quad (1 - e^{-M}) \leq \sum_{r_1, r_2, \dots, r_n \in \mathbb{N}} \prod_{i=1}^n ((1-q)^{r_i} \varrho F_M(r_{i-1}, r_i, u_{i-1})) \leq 1.$$

Moreover, for every  $m \in \{1, 2, \dots, n\}$  and any  $r_1, r_2, \dots, r_{m-1} \in \mathbb{N}$

$$(3.13) \quad (1 - e^{-M}) \leq \sum_{r_m, r_{m+1}, \dots, r_n \in \mathbb{N}} \prod_{i=m}^n ((1-q)^{r_i} \varrho F_M(r_{i-1}, r_i, u_{i-1})) \leq (1 - e^{-M})^{-1}.$$

*Proof.* Lemma 3.4 implies that

$$(1 - e^{-M}) \leq \sum_{r_1, r_2, \dots, r_n \in \mathbb{N}} \prod_{i=1}^n (1-q)^{r_i} \frac{q}{(1-q)\mu} F_M(r_{i-1}, r_i, u_{i-1}) \leq 1.$$

Setting  $\varrho = q/(\mu(1-q))$  we get (3.12). Properties of  $\mu$  (see (3.8)) imply (3.11).

To show (3.13) we first notice that applying (3.12) to the walk that starts at  $x_{m-1}$ , never returns to  $x_{m-1}$ , and reaches  $x_n$  (in the notation of Lemma 3.1) we get

$$(3.14) \quad (1 - e^{-M}) \leq \sum_{r_m, r_{m+1}, \dots, r_n \in \mathbb{N}} \left( (1-q)^{r_m} \varrho F_M(0, r_m, 0) \times \prod_{i=m+1}^n (1-q)^{r_i} \varrho F_M(r_{i-1}, r_i, u_{i-1}) \right) \leq 1.$$

Moreover, by (3.5),

$$F_M(0, r_m, 0) \leq F_M(r_{m-1}, r_m, u_{m-1}) \leq \frac{F_M(0, r_m, 0)}{1 - e^{-M}}.$$

Thus, we can replace  $F_M(0, r_m, 0)$  with  $F_M(r_{m-1}, r_m, u_{m-1})$  in (3.14) at the expense of an extra factor in the right-hand side and obtain (3.13).  $\square$

From Corollary 3.5 we see that for each  $\varrho \in (0, \infty)$  measures  $\Pi_n^\varrho$  on  $\mathbb{N}^n$ ,  $n \in \mathbb{N}$ , defined by

$$\Pi_n^\varrho(\{r_1, r_2, \dots, r_n\}) := \prod_{i=1}^n (1-q)^{r_i} \varrho F_M(r_{i-1}, r_i, u_{i-1}), \quad (r_1, r_2, \dots, r_n) \in \mathbb{N}^n,$$

form an “almost” consistent family of “almost” probability measures.

We can sharpen (3.11) as follows.

**Proposition 3.6.** *Let  $q$  be defined as in Corollary 3.5. Then*

$$\lim_{\varrho \rightarrow 0} \varrho \log(1/q) = 2(e^M - 1) \quad \text{and} \quad \lim_{M \rightarrow \infty} e^{-M} \log(1/q) = 2/\varrho.$$

*Proof.* In view of (3.11) we only need to show that for every  $\varepsilon > 0$  there is  $\varrho_0 > 0$  such that  $\varrho \log(1/q) \leq 2(1 + \varepsilon)(e^M - 1)$  for all  $\varrho \in (0, \varrho_0)$ . Assume the contrary, i.e. that there is  $\varepsilon \in (0, 1/2)$  such that for every  $\varrho_0 > 0$  there is  $\varrho \in (0, \varrho_0)$ , for which  $\varrho \log(1/q) > 2(1 + \varepsilon)(e^M - 1)$ . Then

$$\sum_{r=1}^{\infty} \frac{\varrho(1-q)^r}{2(e^M - 1)r} > 1 + \varepsilon.$$

We shall show that this contradicts to the fact that the total mass of  $\Pi_n^\varrho$  is bounded above by 1 uniformly in  $n \in \mathbb{N}$  and  $\varrho \in (0, \infty)$ . Recall that for all  $\ell \geq 0$ ,  $r \geq 1$  and  $u \in [0, 1]$

$$F(\ell, r, u) \geq F_M(0, r, 0) = \frac{1}{2re^M} = \frac{1 - e^{-M}}{2r(e^M - 1)}$$

and that, by (3.6), for all  $\ell, r > R$ , where  $R$  is sufficiently large,

$$F_M(\ell, r, u) \geq \frac{(1 - \varepsilon/4)}{2r(e^M - 1)}.$$

Given  $r_1, r_2, \dots, r_n$ , we split the set of indices into  $A_R := \{i \in \{1, 2, \dots, n\} : \min\{r_{i-1}, r_i\} \leq R\}$  and  $\bar{A}_R := \{1, 2, \dots, n\} \setminus A_R$ . Then

$$\begin{aligned} \Pi_n^\varrho(\{r_1, \dots, r_n\}) &= \prod_{i=1}^n \varrho(1-q)^{r_i} F_M(r_{i-1}, r_i, u_{i-1}) = \\ &= \prod_{i \in A_R} \varrho(1-q)^{r_i} F_M(r_{i-1}, r_i, u_{i-1}) \times \prod_{i \in \bar{A}_R} \varrho(1-q)^{r_i} F_M(r_{i-1}, r_i, u_{i-1}) \geq \\ &= (1 - e^{-M})^{|A_R|} (1 - \varepsilon/4)^{n - |A_R|} \prod_{i=1}^n \frac{\varrho(1-q)^{r_i}}{2(e^M - 1)r_i}. \end{aligned}$$

Choose  $\delta > 0$  small enough to have  $(1 - e^{-M})^\delta \geq 1 - \varepsilon/4$ . Then

$$\begin{aligned} 1 &\geq \Pi_n^\varrho(|A_R| \leq \delta n) = \sum_{|A_R| \leq \delta n} \prod_{i=1}^n \varrho(1-q)^{r_i} F_M(r_{i-1}, r_i, u_{i-1}) \\ &\geq (1 - \varepsilon/4)^{2n} \sum_{|A_R| \leq \delta n} \prod_{i=1}^n \frac{\varrho(1-q)^{r_i}}{2(e^M - 1)r_i} \\ &= (1 - \varepsilon/4)^{2n} \left( \sum_{r_1, \dots, r_n \in \mathbb{N}} \prod_{i=1}^n \frac{\varrho(1-q)^{r_i}}{2(e^M - 1)r_i} - \sum_{|A_R| > \delta n} \prod_{i=1}^n \frac{\varrho(1-q)^{r_i}}{2(e^M - 1)r_i} \right) \\ &\geq (1 - \varepsilon/4)^{2n} (1 + \varepsilon)^n - (1 - \varepsilon/4)^{2n} \sum_{|A_R| > \delta n} \prod_{i=1}^n \frac{\varrho(1-q)^{r_i}}{2(e^M - 1)r_i} \\ &\geq (1 + \varepsilon/4)^n - (1 - \varepsilon/4)^{2n} \sum_{|A_R| > \delta n} \prod_{i=1}^n \frac{\varrho(1-q)^{r_i}}{2(e^M - 1)r_i}. \end{aligned}$$

To get a contradiction, it is enough to show that the last sum is bounded uniformly in  $n$ . Such a bound is easily obtained from basic large deviations for i.i.d. Bernoulli random variables. Notice that by (3.11) there is a constant  $C$ ,  $1 - e^{-M} \leq C \leq 1$ , such that  $C \varrho(1-q)^r / (2r(e^M - 1))$ ,  $r \in \mathbb{N}$ , is a probability distribution on  $\mathbb{N}$ . Consider a sequence of i.i.d. random variables  $(Y_i)_{i \in \mathbb{N}}$  with this distribution. Then  $P(Y_i \leq R) \leq C \varrho R / (2(e^M - 1)) \rightarrow 0$  as  $\varrho \rightarrow 0$ . Thus, for an arbitrary  $c > 0$  we can choose  $\varrho$  small enough so that for all sufficiently large  $n$

$$P\left(\sum_{i=1}^n \mathbb{1}_{\{Y_i \leq R\}} > \frac{\delta n}{2}\right) \leq e^{-cn}.$$

Since  $\{|A_R| > \delta n\} \subset \{\sum_{i=1}^n \mathbb{1}_{\{r_i \leq R\}} > \delta n/2\}$ ,

$$\sum_{|A_R| > \delta n} \prod_{i=1}^n \frac{\varrho(1-q)^{r_i}}{2(e^M - 1)^{r_i}} \leq C^{-n} P\left(\sum_{i=1}^n \mathbb{1}_{\{Y_i \leq R\}} > \frac{\delta n}{2}\right) \leq (Ce^c)^{-n} \leq 1$$

for  $c > -\log(1 - e^{-M})$  and all large  $n$ , and we are done.  $\square$

We shall need the following comparison lemma. The notation  $\leq_{\text{st}}$  (resp.  $\geq_{\text{st}}$ ) means “stochastically smaller” (resp. “stochastically larger”), where we use the usual stochastic order (see, for example, [SS07, Sec. 6.B]).

**Lemma 3.7.** *Let  $(R_1, R_2, \dots, R_n)$  be distributed according to the probability measure  $\tilde{\Pi}_n^\varrho(\cdot) := \Pi_n^\varrho(\cdot) / \Pi_n^\varrho(\mathbb{N}^n)$ .*

(a) *For  $K = K(M) = (1 - e^{-M})^{-2}$*

$$(R_1, R_2, \dots, R_n) \leq_{\text{st}} (Y_1, Y_2, \dots, Y_n),$$

*where  $(Y_i)_{1 \leq i \leq n}$  are i.i.d.,  $Y_i \in \mathbb{N}$ , and for all  $x \in \mathbb{N}$*

$$(3.15) \quad P(Y_1 \geq x) = 1 \wedge \left( K \sum_{r=x}^{\infty} \frac{\varrho(1-q)^r}{r} \right).$$

(b) *For  $\varkappa = 1/K(M) = (1 - e^{-M})^2$*

$$(R_1, R_2, \dots, R_n) \geq_{\text{st}} (Z_1, Z_2, \dots, Z_n),$$

*where  $(Z_i)_{1 \leq i \leq n}$  are i.i.d.,  $Z_i \in \mathbb{N} \cup \{0\}$ , and for all  $x \in \mathbb{N}$*

$$(3.16) \quad P(Z_1 \geq x) = \varkappa \sum_{r=x}^{\infty} \frac{\varrho(1-q)^r}{r}.$$

*Proof.* Since proofs of both parts are very similar, we prove only part (a). By [SS07, Th. 6.B.B], it is enough to check that there is  $K$  such that for all  $m \geq 1$  and  $r_1, \dots, r_{m-1} \in \mathbb{N}$

$$[R_m \mid R_1 = r_1, \dots, R_{m-1} = r_{m-1}] \leq_{\text{st}} Y_1,$$

where for  $m = 1$  we agree to drop the conditioning. By the definition of  $\tilde{\Pi}_n^\varrho$ , (3.13), and (3.5) we have for each  $x \in \mathbb{N}$

$$\begin{aligned}
& \tilde{\Pi}_n^\varrho(R_m \geq x \mid R_1 = r_1, \dots, R_{m-1} = r_{m-1}) \\
&= \frac{\sum_{r_m \geq x} \sum_{r_{m+1}, \dots, r_n \in \mathbb{N}} \prod_{i=m}^n \varrho(1-q)^{r_i} F_M(r_{i-1}, r_i, u_{i-1})}{\sum_{r_m, \dots, r_n \in \mathbb{N}} \prod_{i=m}^n \varrho(1-q)^{r_i} F_M(r_{i-1}, r_i, u_{i-1})} \\
&\leq 1 \wedge \left( (1 - e^{-M})^{-2} \sum_{r_m \geq x} \varrho(1-q)^{r_m} F_M(r_{m-1}, r_m, u_{m-1}) \right) \\
&\leq 1 \wedge \left( (1 - e^{-M})^{-2} \sum_{r_m \geq x} \frac{\varrho(1-q)^{r_m}}{2r_m(e^M - 1)} \right) \leq 1 \wedge \left( K(M) \sum_{r \geq x} \frac{\varrho(1-q)^r}{r} \right). \quad \square
\end{aligned}$$

**Remark 3.8.** Observe that due to Proposition 3.6  $EZ_1 \rightarrow \infty$  when  $\varrho \rightarrow 0$  or  $M \rightarrow \infty$ .

The next corollary follows the last lemma by Cramér’s theorem.

**Corollary 3.9.** For  $\theta_0 \in (0, 1]$ , for each  $H > EY_1$  (see (3.15)), and each  $h < EZ_1$  (see (3.16)) there exist strictly positive  $c(H)$  and  $c'(h)$  such that for all  $n$  large and  $\theta \in [\theta_0, 1]$

$$\tilde{\Pi}_n^\varrho \left( \sum_{1 \leq i \leq \theta n} r_i > H\theta n \right) \leq e^{-c(H)\theta n} \text{ and } \tilde{\Pi}_n^\varrho \left( \sum_{1 \leq i \leq \theta n} r_i < h\theta n \right) \leq e^{-c'(h)\theta n}.$$

Finally, we are ready to convert information about configurations under  $\tilde{\Pi}_n^\varrho$  to information under  $Q_{0,y}$ . For  $1 \leq n+1 \leq k \leq y$  let

$$\begin{aligned}
A_{n,k} &:= \left\{ (r_1, r_2, \dots, r_k) \in \mathbb{N}^k : \sum_{i=1}^n r_i < y, \sum_{i=1}^{n+1} r_i \geq y \right\}; \\
(3.17) \quad A_n &:= \left\{ (r_1, r_2, \dots, r_{n+1}) \in \mathbb{N}^{n+1} : \sum_{i=1}^{n+1} r_i = y \right\} \subset A_{n,n+1}.
\end{aligned}$$

**Lemma 3.10.** For each  $H > EY_1$  and  $h < EZ_1$  there exist strictly positive  $c_1(H)$  and  $c'_1(h)$  such that for all  $y$  large

$$\begin{aligned}
(1-q)^y \sum_{A_n} \prod_{i=1}^{n+1} \varrho F_M(r_{i-1}, r_i, u_{i-1}) &\leq e^{-c_1(H)y} \quad \text{for all } n \in [0, y/H]; \\
(1-q)^y \sum_{A_n} \prod_{i=1}^{n+1} \varrho F_M(r_{i-1}, r_i, u_{i-1}) &\leq e^{-c'_1(h)y} \quad \text{for all } n \in [y/h, y].
\end{aligned}$$

*Proof.* We shall start with the second inequality. If  $h \leq 1$  then there is nothing to prove as  $[y/h, y]$  is empty. Assume that  $1 < h < EZ_1$  and apply Corollary 3.9 with  $\theta_0 = y/h$ ,  $y$  in place of  $n$ , and  $n$  in place of  $\theta n$ . As  $A_{n,y} \subset \{(r_1, \dots, r_y) \in \mathbb{N}^y : \sum_{i=1}^n r_i < hn\}$ , we get that

for all  $n \in [y/h, y)$

$$(3.18) \quad e^{-c'(h)y/h} \geq e^{-c'(h)n} \geq \sum_{A_{n,y}} \prod_{i=1}^y \varrho(1-q)^{r_i} F_M(r_{i-1}, r_i, u_{i-1}).$$

If  $y \geq n+2$  then we write the right-hand side of the above inequality as

$$\sum_{A_{n,n+1}} \prod_{i=1}^{n+1} \varrho(1-q)^{r_i} F_M(r_{i-1}, r_i, u_{i-1}) \times \left( \sum_{r_{n+2}, \dots, r_y \in \mathbb{N}} \prod_{i=n+2}^y \varrho(1-q)^{r_i} F_M(r_{i-1}, r_i, u_{i-1}) \right),$$

and apply (3.13) to the last summation. Thus, for all  $n \in [y/h, y)$  we have

$$\begin{aligned} e^{-c'(h)y/h} &\geq (1 - e^{-M}) \sum_{A_{n,n+1}} \prod_{i=1}^{n+1} \varrho(1-q)^{r_i} F_M(r_{i-1}, r_i, u_{i-1}) \\ &\geq (1 - e^{-M}) \sum_{A_n} \prod_{i=1}^{n+1} \varrho(1-q)^{r_i} F_M(r_{i-1}, r_i, u_{i-1}) \\ &= (1 - e^{-M})(1-q)^y \sum_{A_n} \prod_{i=1}^{n+1} \varrho F_M(r_{i-1}, r_i, u_{i-1}), \end{aligned}$$

where the last inequality is due to the inclusion  $A_n \subset A_{n,n+1}$ .

The proof of the first inequality is similar. Notice that for all  $n \in [0, y/H)$  we have  $A_{n,y} \subset \{(r_1, \dots, r_y) \in \mathbb{N}^y : \sum_{i=1}^{\lfloor y/H \rfloor + 1} r_i \geq H(y/H)\}$ . Again by Corollary 3.9

$$e^{-c(H/2)y/H} \geq \sum_{A_{n,y}} \prod_{i=1}^y \varrho(1-q)^{r_i} F_M(r_{i-1}, r_i, u_{i-1}), \quad n \in [0, y/H).$$

The rest of the proof follows the proof of the second inequality.  $\square$

**Lemma 3.11.** *Let  $h$  be as in Lemma 3.10. There exists a strictly positive  $c_2 = c_2(\varrho, M)$  such that for  $y$  large*

$$(1-q)^y \sum_{n < y/h} \sum_{A_n} \prod_{i=1}^{n+1} \varrho F_M(r_{i-1}, r_i, u_{i-1}) \geq c_2.$$

*Proof.* We start with (3.18) and sum up over  $n \in [y/h, y)$ . The number of terms in this summation does not exceed  $y$ , therefore, for all sufficiently large  $y$  the sum is less than  $(1 - e^{-M})/2$ . Since the sum over  $n \in [0, y)$  is equal to  $\Pi_y^{\varrho}(\mathbb{N}^y) \geq (1 - e^{-M})$ , we conclude that

$$\frac{1}{2}(1 - e^{-M}) \leq \sum_{n < y/h} \sum_{A_{n,y}} \prod_{i=1}^y \varrho(1-q)^{r_i} F_M(r_{i-1}, r_i, u_{i-1}).$$

Just as in the previous proof, if  $y \geq n+2$  then we perform first the summation over  $r_{n+2}, \dots, r_y \in \mathbb{N}$  and apply (3.13) to get

$$\frac{1}{2}(1 - e^{-M}) \leq (1 - e^{-M})^{-1} \sum_{n < y/h} \sum_{A_{n,n+1}} \prod_{i=1}^{n+1} \varrho(1-q)^{r_i} F_M(r_{i-1}, r_i, u_{i-1}).$$

Next, we replace  $F_M(r_n, r_{n+1}, u_n)$  with its upper bound  $F_M(r_n, y - \sum_{i=1}^n r_i, u_n)$  and sum over  $r_{n+1}$  from  $y - \sum_{i=1}^n r_i$  to infinity. We obtain

$$\begin{aligned} \frac{1}{2}(1 - e^{-M}) &\leq \frac{1}{q(1 - e^{-M})} \sum_{n < y/h} \sum_{A_n} \prod_{i=1}^{n+1} \varrho(1 - q)^{r_i} F_M(r_{i-1}, r_i, u_{i-1}) \\ &= \frac{(1 - q)^y}{q(1 - e^{-M})} \sum_{n < y/h} \sum_{A_n} \prod_{i=1}^{n+1} \varrho F_M(r_{i-1}, r_i, u_{i-1}). \end{aligned}$$

This gives the desired statement with  $c_2 = q(1 - e^{-M})^2$ .  $\square$

**Corollary 3.12.** *In (3.7),*

$$Z_{0,y}^{-1}(1 - p)^y \left( \frac{e^M}{\varrho} + 1 \right) \leq c_2^{-1}(1 - q)^y,$$

where  $c_2$  is the same as in Lemma 3.11.

**3.3. Final step: construction of  $\Omega_y^1$  and  $\Omega_y^2$ .** Throughout this subsection we suppose that for a given environment  $\omega$ , the occupied sites in  $(0, y)$  are  $\{x_1, x_2, \dots, x_n\}$ , where  $0 = x_0 < x_1 < \dots < x_n < y := x_{n+1}$ . As before, we set  $r_i = x_i - x_{i-1}$ ,  $i = 1, 2, \dots, n + 1$ .

Let  $\varrho = p/(1 - p)$ . Fix an  $\varepsilon > 0$  and set  $h = e^{2(1-\varepsilon/2)(e^M-1)/\varrho}$ . We claim that  $h < EZ_1$  for all sufficiently small  $\varrho$  ( $M$  is fixed). Indeed, by (3.16) and Proposition 3.6

$$EZ_1 = \frac{\varkappa(M)\varrho(1 - q(\varrho))}{q(\varrho)}, \text{ where } \frac{1}{q(\varrho)} = e^{2(e^M-1)(1+o(\varrho))/\varrho} \text{ as } \varrho \rightarrow 0.$$

Let  $\mathcal{A}_n := \{\omega \in \Omega : N(\omega, y) = n, (R_1(\omega), R_2(\omega), \dots, R_n(\omega)) \in A_n\}$ , where  $A_n$  was defined in (3.17). Then by Lemma 3.10 and Corollary 3.12 we have that for all sufficiently small  $p$  and large  $y$  ( $c_1$  and  $c_2$  do not depend on  $y$ )

$$(3.19) \quad Q_{0,y}(\cup_{n > y/h} \mathcal{A}_n) \leq \frac{y}{c_2} e^{-c_1 y}.$$

Now we are ready to construct  $\Omega_y^1$ . Let  $\Omega_y^1 = \cup_{n \leq y/h} \mathcal{A}_n$ . The bound (3.19) implies that  $Q_{0,y}(\Omega_y^1) \geq 1/2$  for all sufficiently large  $y$ . Thus, (L1) is satisfied. Let  $\omega \in \Omega_y^1$ . By Lemma A.2 and Cauchy-Schwarz inequality,

$$E_{Q_{0,y}^\omega} \tau_y \geq \frac{1}{3} \sum_{i=1}^{n+1} r_i^2 \geq \frac{y^2}{3(n+1)} \geq \frac{y^2}{3(y/h+1)} \geq \frac{hy}{3} = \frac{y}{3} e^{(1-\varepsilon)K(p,M)}.$$

This gives us (L2) and the desired lower bound (3.2).

The upper bound is somewhat more involved. Lemma A.2 provides us with the following estimate:

$$E_{Q_{0,y}^\omega} \tau_y \leq \frac{1}{3(1 - e^{-M})} \sum_{j=1}^{n+1} r_j^2.$$

We set  $\Omega_y^2 = \cup_{n \leq y/h} \mathcal{B}_n$ , where

$$\mathcal{B}_n = \left\{ \omega \in \Omega : N(\omega, y) = n, \sum_{i=1}^{n+1} R_i(\omega) = y, \sum_{i=1}^{n+1} R_i^2(\omega) \leq \frac{C_3 n}{\log^2(1/(1 - q))} \right\},$$

and  $C_3$  is a sufficiently large constant, which we shall determine later (see Lemma 3.13). The term  $\log(1/(1-q))$  gives us the right scaling, since under the probability measure  $\tilde{\Pi}_n^\varrho$  the expected “gap” size is roughly of order  $1/q \sim 1/\log(1/(1-q))$  as  $q \rightarrow 0$ . We shall show that with this scaling the constant  $C_3$  can indeed be chosen uniformly over all small  $p$  and large  $M$ .

At first, we check that  $\Omega_y^2$  satisfies (U2). Let  $\omega \in \Omega_y^2$ . Then  $n \leq y/h$  and

$$E_{Q_{0,y}^\omega} \tau_y \leq \frac{C_3 n}{3(1-e^{-M}) \log^2(1/(1-q))} \leq \frac{C_3 y}{3(1-e^{-M}) q^2 h}.$$

By Proposition 3.6,  $q^{-1} \leq \exp(2(1+\varepsilon/4)(e^M - 1)/\varrho)$  for all sufficiently small  $\varrho$ . Substituting the expressions for  $h$  and  $\varrho$  we get the desired upper bound

$$E_{Q_{0,y}^\omega} \tau_y \leq \frac{C_3 y}{3(1-e^{-M})} e^{(1+\varepsilon)K(p,M)}.$$

Our last task is to establish (U1). Notice that  $\Omega \setminus \Omega_y^2 \subset (\cup_{n>y/h} \mathcal{A}_n) \cup (\cup_{n<y/H} \mathcal{A}_n) \cup (\cup_{y/H < n \leq y/h} \mathcal{D}_n)$ , where  $h$  and  $H$  are chosen as in Lemma 3.10, and

$$\mathcal{D}_n = \left\{ \omega \in \Omega : N(\omega, y) = n, \sum_{i=1}^{n+1} R_i(\omega) = y, \sum_{i=1}^{n+1} R_i^2(\omega) > \frac{C_3 n}{\log^2(1/(1-q))} \right\}.$$

Therefore, by (3.19), Lemma 3.10, and Corollary 3.12,

$$(3.20) \quad Q_{0,y}(\Omega \setminus \Omega_y^2) \leq \frac{y}{c_2} e^{-c'_1(h)y} + \frac{y}{c_2} e^{-c_1(H)y} + y \max_{y/H \leq n \leq y/h} Q_{0,y}(\mathcal{D}_n).$$

Again we estimate first probabilities  $\tilde{\Pi}_n^\varrho$  of the relevant events.

**Lemma 3.13.** *For  $\theta_0 \in (0, 1]$  there are  $c_3, C_3 > 0$  such that for all sufficiently large  $n$  and  $\theta \in [\theta_0, 1]$*

$$\tilde{\Pi}_n^\varrho \left( \sum_{i=1}^{\theta n} R_i^2 > \frac{C_3 \theta n}{\log^2(1/(1-q))} \right) \leq e^{-c_3 \sqrt{\theta n}}.$$

*Proof.* By part (a) of Lemma 3.7

$$\tilde{\Pi}_n^\varrho \left( \sum_{i=1}^{\theta n} R_i^2 > \frac{C_3 \theta n}{\log^2(1/(1-q))} \right) \leq P \left( \sum_{i=1}^{\theta n} Y_i^2 > \frac{C_3 \theta n}{\log^2(1/(1-q))} \right).$$

Thus we need a large deviations upper bound for a sequence of i.i.d. random variables  $Y_i^2 \log^2(1/(1-q))$ ,  $i \in \mathbb{N}$ , which have sub-exponential tails: there are  $\ell \in \mathbb{N}$ ,  $\varrho_0, M_0 > 0$  such that for all  $x \geq \ell$  and  $(\varrho, M) \in [0, \varrho_0] \times [M_0, \infty)$

$$\begin{aligned} P(Y_i^2 \log^2(1/(1-q)) \geq x) &= K(M) \sum_{r \geq \sqrt{x}/\log(1/(1-q))} \frac{\varrho(1-q)^r}{r} \\ &\leq \frac{K(M) \varrho(1-q)^{-\sqrt{x}/(\log(1-q))}}{\sqrt{x}/\log(1/(1-q))} \sum_{r \geq 0} (1-q)^r \leq \frac{K(M) \varrho}{\sqrt{x_0}} e^{-\sqrt{x}} q^{-1} \log \frac{1}{1-q} \leq e^{-\sqrt{x}}. \end{aligned}$$

The statement of the lemma now follows from Lemma A.3.  $\square$

It only remains to convert (as we have done before) the previous result into a bound on  $Q_{0,y}$  probability.



**Lemma 3.14.** *Let  $H$  be chosen as in Lemma 3.10. There is strictly positive  $c_4 = c_4(H)$  such that for all  $n \in [y/H, y)$*

$$(1-q)^y \sum_{D_n} \prod_{i=1}^{n+1} \varrho F_M(r_{i-1}, r_i, u_{i-1}) \leq e^{-c_4 \sqrt{y}}.$$

*Proof.* For  $1 \leq n+1 \leq k \leq y$  let

$$D_{n,k} := \left\{ (r_1, \dots, r_k) \in \mathbb{N}^k : \sum_{i=1}^n r_i < y, \sum_{i=1}^{n+1} r_i \geq y, \sum_{i=1}^{n+1} r_i^2 > \frac{C_3 n}{\log^2(1/(1-q))} \right\}$$

$$D_n := \left\{ (r_1, r_2, \dots, r_{n+1}) \in \mathbb{N}^{n+1} : \sum_{i=1}^{n+1} r_i = y, \sum_{i=1}^{n+1} r_i^2 > \frac{C_3 n}{\log^2(1/(1-q))} \right\}.$$

Since  $D_{n,y} \subset \{(r_1, \dots, r_y) \in \mathbb{N}^y : \sum_{i=1}^{n+1} r_i^2 > C_3 n \log^{-2}(1/(1-q))\}$  and  $n \geq y/H$ , we have by Lemma 3.13 that

$$e^{-c_3 \sqrt{y/H}} \geq e^{-c_3 \sqrt{n}} \geq \sum_{D_{n,y}} \prod_{i=1}^y \varrho (1-q)^{r_i} F_M(r_{i-1}, r_i, u_{i-1}).$$

The rest of the proof follows the one of Lemma 3.10 with  $D_{n,y}$  and  $D_n$  in place of  $A_{n,y}$  and  $A_n$ .  $\square$

Lemma 3.14 and Corollary 3.9 imply that

$$\max_{y/H \leq n \leq y/h} Q_{0,y}(\mathcal{D}_n) \leq c_2^{-1} e^{-c_4 \sqrt{y}}.$$

Together with (3.20) this estimate establishes (U1).

## APPENDIX A. PROOFS OF TECHNICAL LEMMAS

*Proof of Proposition 2.1.* Let  $u_0 = u_n = 0$ , and  $u_k = E^k(\tau_n; \tau_n < \tau_0)$ . Then

$$\begin{aligned} u_k &= E^k(\tau_n; \tau_n < \tau_0) \\ &= \frac{1}{2} E^{k+1}(\tau_n + 1; \tau_n < \tau_0) + \frac{1}{2} E^{k-1}(\tau_n + 1; \tau_n < \tau_0) \\ &= \frac{1}{2} u_{k+1} + \frac{1}{2} u_{k-1} + \frac{1}{2} P^{k+1}(\tau_n < \tau_0) + \frac{1}{2} P^{k-1}(\tau_n < \tau_0) \\ &= \frac{1}{2} u_{k+1} + \frac{1}{2} u_{k-1} + \frac{1}{2} \frac{k+1}{n} + \frac{1}{2} \frac{k-1}{n} = \frac{1}{2} u_{k+1} + \frac{1}{2} u_{k-1} + \frac{k}{n}. \end{aligned}$$

Denoting by  $\Delta u_k$  the discrete Laplacian of  $u_k$ , we get that

$$-\frac{1}{2} \Delta u_k = \frac{k}{n}, \quad k = 0, 1, 2, \dots, n.$$

The continuous analog is  $-\Delta u = 2x$ ,  $u(0) = u(1) = 0$ ,  $x \in [0, 1]$ , and it is easy to solve:  $u(x) = x(1-x)(1+x)/3$ . From the scaling property of the random walk we conclude that  $u_k = (3n)^{-1} k(n-k)(n+k)$ . It is also easy to check directly that this expression indeed solves the equation  $-\frac{1}{2} \Delta u_k = k/n$ ,  $k = 0, 1, 2, \dots, n$ , and satisfies the boundary conditions.  $\square$

*Proof of Proposition 2.2.* Notice that we can omit  $\mathbb{1}_{\{\tau_y < \infty\}}$  from (1.5), (1.6), and (1.16), since on the event  $\{\tau_y = \infty\}$  the exponential function in the integrand vanishes  $P$ -a.s.. By the strong Markov property, for every  $\lambda \geq 0$  and  $y \in \mathbb{N}$  we can write

$$\begin{aligned} \frac{1}{y} \log E \left( e^{-\sum_{n=0}^{y-1} (\lambda + V(S_n, \omega))} \right) &= \frac{1}{y} \sum_{k=0}^{y-1} \log E^k \left( e^{-\sum_{n=0}^{\tau_{k+1}-1} (\lambda + V(S_n, \omega))} \right), \\ \frac{E_{Q_y^\omega} \tau_y}{y} &= \frac{1}{y} \sum_{k=0}^{y-1} E_{Q_{k+1}^{\omega, k}}(\tau_{k+1}). \end{aligned}$$

The ergodic theorem implies that

$$\begin{aligned} \alpha_{\lambda+V}(1) &= -\mathbb{E}(\log E \left( e^{-\sum_{n=0}^{\tau_1-1} (\lambda + V(S_n, \omega))} \right)), \\ \frac{1}{v_{\text{que}}} &= \mathbb{E}(E_{Q_1^\omega}(\tau_1)) = \mathbb{E}(E^\omega(\tau_1 | \tau_1 < \infty)). \end{aligned}$$

The function  $\alpha_{\lambda+V}(1)$  is concave and monotone non-decreasing (see [Ze98, p. 272]). Taking the right derivative of  $\alpha_{\lambda+V}(1)$  at  $\lambda = 0$  we obtain the last statement of proposition. But we still need to show that  $\mathbb{E}(E^\omega(\tau_1 | \tau_1 < \infty)) < \infty$ . Let  $\Lambda(t) = -\log \mathbb{E}(e^{-tV(0, \cdot)})$  and  $\ell(x) = \sum_{n=0}^{\tau_1-1} \mathbb{1}_{\{S_n=x\}}$ . Then, since  $P^\omega(\tau_1 < \infty) \geq e^{-V(0, \omega)}/2$ ,

$$\begin{aligned} \mathbb{E}(E^\omega(\tau_1 | \tau_1 < \infty)) &\leq \mathbb{E} \left( 2e^{V(0, \cdot)} E \left( \tau_1 e^{-\sum_{n=0}^{\tau_1-1} V(S_n, \cdot)} \right) \right) \leq 2E \left( \tau_1 e^{-\sum_{x < 0} \Lambda(\ell(x))} \right) \\ &\leq 2P(\tau_1 < \tau_{-1}) + 2 \sum_{k=1}^{\infty} E \left( \tau_1 e^{-k\Lambda(1)}; \tau_{-k} < \tau_1 < \tau_{-(k+1)} \right) \\ &\leq 1 + 2 \sum_{k=1}^{\infty} e^{-k\Lambda(1)} E(\tau_1; \tau_1 < \tau_{-(k+1)}) \stackrel{\text{Prop. 2.1}}{<} \infty. \quad \square \end{aligned}$$

*Proof of Lemma 2.4.* The proof is very simple. If  $a_1 = 0$  then a trivial bound is given by the survival probability  $e^{-M}$ . For  $a_1 \in \mathbb{N}$  we have

$$\begin{aligned} P^{\omega, 0}(\tau_1 > \tau_{-a_1} | \tau_1 < \infty) \\ = \frac{P^{\omega, 0}(\tau_1 < \infty | \tau_1 > \tau_{-a_1}) P^{\omega, 0}(\tau_1 > \tau_{-a_1})}{P^{\omega, 0}(\tau_1 < \infty)} = \frac{1}{a_1 + 1} \frac{P^{\omega, -a_1}(\tau_1 < \infty)}{P^{\omega, 0}(\tau_1 < \infty)}. \end{aligned}$$

It is obvious that the last ratio is bounded by  $e^{-M}$  but we shall need an improvement of this estimate for the proof of Lemma 2.5. We have

$$\begin{aligned} \frac{P^{\omega, -a_1}(\tau_1 < \infty)}{P^{\omega, 0}(\tau_1 < \infty)} &= P^{\omega, -a_1}(\tau_0 < \infty) = e^{-M} \left( \frac{1}{2a_1} + \frac{a_1 - 1}{2a_1} P^{\omega, -a_1}(\tau_0 < \infty) \right. \\ &\quad \left. + \frac{1}{2} P^{\omega, -a_1-1}(\tau_0 < \infty | \tau_{-a_1} < \infty) P^{\omega, -a_1-1}(\tau_{-a_1} < \infty) \right) \\ &\leq e^{-M} \left( \frac{1}{2a_1} + \left( 1 - \frac{1}{2a_1} \right) P^{\omega, -a_1}(\tau_0 < \infty) \right). \end{aligned}$$

This gives

$$(A.1) \quad \frac{P^{\omega, -a_1}(\tau_1 < \infty)}{P^{\omega, 0}(\tau_1 < \infty)} \leq \frac{e^{-M}}{2a_1(1 - e^{-M}) + e^{-M}},$$

and the statement of the lemma follows.  $\square$

*Proof of Lemma 2.5.* Suppose  $a_1 = 0$ . Recall that  $\tau_0$  is the time of the first return to 0. We just have to bound the expected number of visits to 0 before  $\tau_1$ :

$$\begin{aligned}
E^\omega(\mathbb{1}_{\{\tau_0 < \tau_1\}} \sum_{n=0}^{\tau_1-1} \mathbb{1}_{\{S_n=0\}} \mid \tau_1 < \infty) &\leq P^{\omega,0}(\tau_0 < \tau_1 \mid \tau_1 < \infty) \\
&\quad + E^\omega(\mathbb{1}_{\{\tau_0 < \tau_1\}} \sum_{n=1}^{\tau_1-1} \mathbb{1}_{\{S_n=0\}} \mid \tau_1 < \infty) \\
&\leq e^{-M} + \frac{P^{\omega,0}(\tau_0 < \tau_1)}{P^{\omega,0}(\tau_1 < \infty)} E^\omega \left( \sum_{n=0}^{\tau_1-1} \mathbb{1}_{\{S_n=0\}} \mathbb{1}_{\{\tau_1 < \infty\}} \right) \\
&\leq e^{-M} + E^\omega \left( \sum_{n=0}^{\tau_1-1} \mathbb{1}_{\{S_n=0\}} \mathbb{1}_{\{\tau_1 < \infty\}} \right) \leq 3e^{-M}.
\end{aligned}$$

We used the following obvious facts:  $P^{\omega,0}(\tau_0 < \tau_1) \leq (2e^M)^{-1}$  and  $P^{\omega,0}(\tau_1 < \infty) \geq (2e^M)^{-1}$ . The last inequality in the multi-line formula above is obtained by considering the Markov chain which starts at 0, gets killed with probability  $(1 - e^{-M})$  after each visit to 0, otherwise goes with equal probabilities to 1 and 0, and always gets absorbed at 1. For such chain the expected time to hit 1 restricted to the event that it reaches 1 is equal to  $2e^{-M}/(2 - e^{-M})^2 \leq 2e^{-M}$ .

Assume now that  $a_1 \in \mathbb{N}$ . Since  $P^{\omega,0}(\tau_1 < \infty) \geq 1/2$ , we can replace the conditioning on the event  $\{\tau_1 < \infty\}$  with the intersection at the cost of factor 2. The time spent in  $I$  before hitting 1 is the sum of three terms: (1)  $\tau_{-a_1}$ ; (2) the total time spent in excursions to  $I$  from  $-a_1$  that end up in  $-a_1$ ; (3) the time needed to get from  $-a_1$  to 1 without returning to  $-a_1$ .

Term (1) is estimated using Lemma 2.2. We get  $Ca_1$  times  $P^{\omega,-a_1}(\tau_1 < \infty)$ . By (A.1) we have the required bound  $Ce^{-M}/(1 - e^{-M})$ . Term (3) is bounded by the product of  $P^{\omega,0}(\tau_{-a_1} < \tau_1)$ ,  $e^{-M}$ , and  $1 + E^{-a_1+1}(\tau_1; \tau_1 < \tau_{-a_1})$ . The latter is again bounded by  $Ca_1$ . Thus, (3) does not exceed  $Ce^{-M}$ . Term (2) is the sum of a random number of durations of excursions. The expected duration of one such excursion is bounded by  $Ca_1$  (Lemma 2.2), and the number of them is at most geometric with expectation  $e^{-M}/(1 - e^{-M})$ . The total is multiplied by  $P^{\omega,0}(\tau_{-a_1} < \tau_1)$ , since to have such excursion the path has to get to  $-a_1$  before hitting 1. The strong Markov property implies the desired bound. Adding the three terms we get the statement of the lemma.  $\square$

*Proof of Lemma 2.6.* This is a rough bound, which is sufficient for our purposes. Just as in the proof of Lemma 2.5, we replace the conditioning on the event  $\{\tau_1 < \infty\}$  by the intersection with  $\{\tau_1 < \infty\}$  at a cost of factor  $2e^M$  (here we can not exclude the case  $a_1 = 0$ ). The time spent in  $I_j$ ,  $j \in \mathbb{N}$ , is equal to the sum of durations of excursions contained in  $I_j$  from  $-a_{j+1}$  and  $-a_j$  as well as crossings between the end points of  $I_j$ . The total number of such excursions and crossings is again at most geometric with expectation  $e^{-M}/(1 - e^{-M})$  and the expected duration is bounded by  $C(a_{j+1} - a_j)$  due to Lemma 2.2. But to have a chance to undergo at least one such excursion or crossing the walk has to reach  $-a_j$  prior to hitting 1, survive at least one visit to each  $-a_i$ ,  $1 \leq i < j$ , and after completing the excursions reach 1 before returning to  $-a_j$  surviving the final run through  $\{-a_i, j \geq i \geq 1\}$ . Thus, the expectation, which we want to estimate is bounded by

$$\frac{C(a_{j+1} - a_j)}{1 - e^{-M}} \frac{e^{-M(2j-1)}}{(a_j + 1)^2}.$$

Replacing  $2j - 1$  with  $j$  and  $a_j + 1$  with  $a_1 + 1$  we obtain the desired upper bound.  $\square$

*Proof of Lemma 3.1.* By simple gambler's ruin considerations we have

$$u_1 = \frac{e^{-V(0,\omega)}}{2r_1} = e^{M-V(0,\omega)} F_M(0, r_1, 0),$$

and for  $n > 1$

$$u_n = e^{-M} \left( \frac{1}{2r_n} + \frac{u_{n-1}u_n}{2r_{n-1}} + u_n \left( 1 - \frac{1}{2r_n} - \frac{1}{2r_{n-1}} \right) \right).$$

Solving the last equation for  $u_n$  we get for  $n > 1$

$$u_n = \frac{e^{-M}}{2r_n} \left( 1 - e^{-M} \left( 1 - \frac{1}{2r_n} - \frac{1 - u_{n-1}}{2r_{n-1}} \right) \right)^{-1} = F_M(r_{n-1}, r_n, u_{n-1})$$

as claimed.  $\square$

**Lemma A.1.** *For every  $p \in [0, 1)$ ,  $M \in [0, \infty)$ ,  $y \in \mathbb{N}$ , and  $\omega \in \Omega$*

$$EQ_{0,y}^\omega \tau_y \leq 1 + 2y^2.$$

*Proof.* Let  $\ell_y(x) = \sum_{n=0}^{\tau_y-1} \mathbb{1}_{\{S_n=x\}}$ . We need to estimate

$$(A.2) \quad EQ_{0,y}^\omega \tau_y = 1 + \sum_{x=1}^{y-1} EQ_{0,y}^\omega \ell_y(x) = 1 + \sum_{x=1}^{y-1} \sum_{m=0}^{\infty} Q_{0,y}^\omega(\ell_y(x) > m).$$

Denote by  $\mathcal{G}_n$  the sigma-algebra generated by the simple random walk up to time  $n$ . Then by the strong Markov property of the simple random walk we have

$$\begin{aligned} Z_{0,y}^\omega Q_{0,y}^\omega(\ell_y(x) > m) &= E \left( e^{-\sum_{n=0}^{\tau_y-1} V(S_n, \omega)} \mathbb{1}_{\{\tau_y < \tau_0, \tau_x^{(m+1)} < \tau_y\}} \right) \\ &\leq E \left( e^{-\sum_{n=0}^{\tau_x-1} V(S_n, \omega)} \mathbb{1}_{\{\tau_x < \tau_0\}} E \left( e^{-\sum_{n=\tau_x}^{\tau_y-1} V(S_n, \omega)} \mathbb{1}_{\{\tau_y < \tau_0, \tau_x^{(m+1)} < \tau_y\}} \mid \mathcal{G}_{\tau_x} \right) \right) \\ &= E \left( e^{-\sum_{n=0}^{\tau_x-1} V(S_n, \omega)} \mathbb{1}_{\{\tau_x < \tau_0\}} E^x \left( e^{-\sum_{n=\tau_x}^{\tau_y-1} V(S_n, \omega)} \mathbb{1}_{\{\tau_y < \tau_0, \tau_x^{(m)} < \tau_y\}} \right) \right) \\ &= E \left( e^{-\sum_{n=0}^{\tau_x-1} V(S_n, \omega)} \mathbb{1}_{\{\tau_x < \tau_0\}} E^x \left( e^{-\sum_{n=\tau_x}^{\tau_y-1} V(S_n, \omega)} \mathbb{1}_{\{\tau_y < \tau_0\}} \mid \tau_x^{(m)} < \tau_y \wedge \tau_0 \right) P^x \left( \tau_x^{(m)} < \tau_y \wedge \tau_0 \right) \right) \\ &= E \left( e^{-\sum_{n=0}^{\tau_x-1} V(S_n, \omega)} \mathbb{1}_{\{\tau_x < \tau_0\}} E^x \left( e^{-\sum_{n=0}^{\tau_y-1} V(S_n, \omega)} \mathbb{1}_{\{\tau_y < \tau_0\}} \right) P^x \left( \tau_x^{(m)} < \tau_y \wedge \tau_0 \right) \right) \\ &= Z_{0,y}^\omega P^x(\tau_x^{(m)} < \tau_y \wedge \tau_0). \end{aligned}$$

Substituting this into (A.2) and taking into account that

$$P^x(\tau_x^{(m)} < \tau_y \wedge \tau_0) = \left( \frac{x-1}{2x} + \frac{y-x-1}{2(y-x)} \right)^m = \left( 1 - \frac{y}{2x(y-x)} \right)^m,$$

we get

$$EQ_{0,y}^\omega \tau_y \leq 1 + \sum_{x=1}^{y-1} \sum_{m=0}^{\infty} \left( 1 - \frac{y}{2x(y-x)} \right)^m = 1 + \frac{2}{y} \sum_{x=1}^{y-1} x(y-x) \leq 1 + 2y^2.$$

$\square$

**Lemma A.2.** *Fix an arbitrary  $p \in [0, 1)$ ,  $M \in [0, \infty)$ ,  $\omega \in \Omega$ , and  $y \geq 1$ . Let  $\mathcal{O} := \{x_1, x_2, \dots, x_n\}$  be the set of all occupied sites in  $(0, y)$ ,  $0 =: x_0 < x_1 < \dots < x_n < x_{n+1} := y$ . Set  $r_j = x_j - x_{j-1}$ ,  $j = 1, 2, \dots, n+1$ . Then*

$$\frac{1}{3} \sum_{j=1}^{n+1} r_j^2 \leq E_{Q_{0,y}^\omega} \tau_y \leq \frac{1}{3(1 - e^{-M})} \sum_{j=1}^{n+1} r_j^2.$$

*Proof.* If  $n = 0$ , i.e.  $\mathcal{O} = \emptyset$ , then the statement follows from the properties of the standard random walk (see Proposition 2.2). Assume that  $n \geq 1$ , and, therefore,  $y \geq n+1 \geq 2$ . The proof is based on the decomposition of random walk paths according to the total number of visits,  $k$ , to  $\mathcal{O}$  before hitting  $y$  and to the order, in which the walk visits points of  $\mathcal{O}$ . For each  $k \geq n$ , such orderings are represented by admissible sequences  $(y_1, y_2, \dots, y_k) \in \mathcal{O}^k$ . For example, for  $n = 3$  and  $k = 9$  a sequence  $(x_1, x_2, x_2, x_3, x_2, x_1, x_2, x_3, x_4)$  is admissible. The set of all admissible sequences of length  $k$  will be denoted by  $\mathcal{Y}_k$  and the set of all random walk paths corresponding to a given admissible  $\mathbf{y}_k \in \mathcal{Y}_k$  by  $W_{\mathbf{y}_k}$ . Define  $\sigma_0 = 0$ ,  $\sigma_i = \inf\{j > \sigma_{i-1} : S_j = y_i\}$ ,  $i \in \mathbb{N}$ . With this notation, we have  $\tau_y = \sum_{i=1}^k (\sigma_i - \sigma_{i-1})$  and

$$(A.3) \quad E_{Q_{0,y}^\omega} \tau_y = \frac{\sum_{k=n}^{\infty} e^{-Mk-V(0)} \sum_{\mathbf{y}_k \in \mathcal{Y}_k} E^0 \left( \sum_{i=1}^k (\sigma_i - \sigma_{i-1}) \mathbb{1}_{W_{\mathbf{y}_k}} \right)}{\sum_{k=n}^{\infty} e^{-Mk-V(0)} \sum_{\mathbf{y}_k \in \mathcal{Y}_k} E^0(\mathbb{1}_{W_{\mathbf{y}_k}})}.$$

It is easy to see from Proposition 2.1 that

$$(A.4) \quad E^0(\tau_n | \tau_n < \tau_0^{(2)}) = \frac{n^2 + 2}{3} \geq \frac{n^2}{3} \text{ and } E^k(\tau_n | \tau_n < \tau_0^{(2)}) = \frac{n^2 - k^2}{3} \leq \frac{n^2}{3}.$$

for all  $n \in \mathbb{N}$  and  $k \in \{1, \dots, n-1\}$ . Since the random walk has to cross every interval  $(x_i, x_{i+1})$ ,  $i = 0, \dots, n$ , at least once, the strong Markov property of the standard random walk, (A.3), and the first inequality in (A.4) immediately give us the claimed lower bound.

We turn now to an upper bound. Using the strong Markov property and the second inequality in (A.4), we can estimate the right-hand side of (A.3) from above by

$$\begin{aligned} & \frac{\left( \sum_{j=1}^{n+1} r_j^2 \right) \sum_{k=n}^{\infty} \left( e^{-Mk-V(0)} \sum_{\mathbf{y}_k \in \mathcal{Y}_k} \left( \max_{1 \leq j \leq n} \sum_{i=1}^k \mathbb{1}_{\{y_i = x_j\}} \right) P^0(W_{\mathbf{y}_k}) \right)}{3 \sum_{k=n}^{\infty} \left( e^{-Mk-V(0)} \sum_{\mathbf{y}_k \in \mathcal{Y}_k} P^0(W_{\mathbf{y}_k}) \right)} \\ & \leq \frac{1}{3} \left( \sum_{j=1}^{n+1} r_j^2 \right) \max_{x \in \mathcal{O}} E_{Q_{0,y}^\omega} \ell_y(x), \end{aligned}$$

where  $\ell_y(x) = \sum_{j=0}^{\tau_y-1} \mathbb{1}_{\{S_j=x\}}$ . The last term can be handled in the same way as in the proof of Lemma A.1. The only difference is that each additional visit to an occupied site adds the factor  $e^{-M}$ , and it is this factor that plays a major role. For every  $x \in \mathcal{O}$  we have

$$E_{Q_{0,y}^\omega} \ell_y(x) = \sum_{m=0}^{\infty} Q_{0,y}^\omega(\ell_y(x) > m) \leq \sum_{m=0}^{\infty} e^{-Mm} P^x(\tau_x^{(m+1)} < \tau_y \wedge \tau_0) \leq \frac{1}{1 - e^{-M}}.$$

This completes the proof.  $\square$

**Lemma A.3.** *Let  $X_i$  be i.i.d. non-negative random variables such that  $P(X_1 \geq x) \leq e^{-\sqrt{x}}$  for all  $x \geq N$ . There exist constants  $K, c \in (0, \infty)$  such that for all  $n$  large*

$$P\left(\sum_{i=1}^n X_i \geq Kn\right) \leq e^{-c\sqrt{n}}.$$

This fact is contained in [Na79, Theorem 1.1, p. 748] but for convenience of the reader we give a short proof.

*Proof.* Fix a small  $\varepsilon > 0$ . Notice that

$$\left\{\sum_{i=1}^n X_i \geq Kn\right\} \subset \left(\bigcup_{i=1}^n \{X_i \geq \varepsilon n\}\right) \cup \left\{\sum_{i=1}^n X_i^{\varepsilon n} \geq Kn\right\}, \quad \text{where } X_i^{\varepsilon n} = X_i \mathbb{1}_{\{X_i \leq \varepsilon n\}}.$$

The probability of the first union is bounded by  $ne^{-\sqrt{\varepsilon n}}$  (for  $\varepsilon n > N$ ), which is less than  $e^{-\sqrt{\varepsilon n/2}}$  for all large  $n$ . It remains to bound the probability of the last event. Since  $e^x \leq 1 + 2x$  for all  $x \in [0, 1]$ , we have  $E(e^{X_i^{\varepsilon n}/(2\sqrt{n})}) \leq 1 + E(X_1)/\sqrt{n}$  for all  $n$  large. The rest follows from the usual Chebyshev bounds: we can choose  $K$  large enough so that for all sufficiently large  $n$

$$P\left(\sum_{i=1}^n X_i^{\varepsilon n} \geq Kn\right) \leq e^{-K\sqrt{n}/2} \left(1 + \frac{E(X_1)}{\sqrt{n}}\right)^n \leq e^{-c\sqrt{n}}, \quad \text{where } c > 0.$$

□

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